Amherst College

Summer 2020 Math Research

Daily Meeting Notes

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Goal of the project: Study dynamics on graphs, specifically small world networks (SWN)

Graphs - Graphs consist of two sets, V and E, where V is the set of vertices and E is the set of edges.For our use, graphs are finite.

Examples of finite graphs:

1. Simple Graphs
2. Multi-graphs (no multi-edges or loops)
3. Pseudo-graphs (graph with loops)

Theorem 1. *1st Fundamental Theorem of Graph Theory: The sum of the degree sequence of vertex v is double the number of edges.*

$\sum_v deg(v) = 2 * (\text{ of edges})$

Metrics

Δ(G)= maximum degree of G

δ(G)= minimum degree of G

<u>Characteristics of Metric Functions:</u>

i. d(u,v) = d(v,u)

ii. d(u,v) + d(v,w) $\geq d(u,w)$

iii. d(u,v) $\geq$ 0 for all u,v and d(u,v) = 0 IFF u=v

Definition 2. A walk on a graph is a sequence of edges that share a vertex. The length of a walk is the number of edges, counting repetitions.

Definition 3. A path is a walk that does not repeat a vertex.

Definition 4. A trail is a walk that does not repeat edges.

Definition 5. Distance on a graph, d(u,v)= length of the shortest path that connects u and v.

Definition 6. Eccentricity, ecc(v) = the greatest distance from v to any other vertex.

- The radius of G, rad(G) is the value of smallest eccentricity.
- The diameter of G, diam(G) is the value of maximum eccentricity.
- The center of G is the set of vertices such that ecc(V)=rad(G)

Types of Graphs

1. Path graph
2. Cycle graph
3. Complete graphs
4. Star graphs
5. Wheel graphs
6. Lollipop graphs
7. Random graphs

Small World Networks (SWN) are a specific type of graph with the following features:

- Plenty of shared neighbors between vertices
- Relatively few edges compared to the vertices in a graph
- Distances between vertices are relatively small
- Reasonable amount of clustering

Characteristic path length, L(G):

L(G)= $\frac{\sum_{u,v \epsilon V(G)} d(u,v)}{\binom{n}{2}}$

Clustering Coefficient:

Definition 7. Let v ϵ V.

CC(v) $=\frac{E<N[v]>}{E(k_{deg(v)+1}}$

The numerator can be defined as the number of edges in the neighbors of v (including v) of the induced sub-graph (an induced sub-graph is a sub-graph in which we use all possible edges).

The denominator can be defined as the number of edges in the complete graph of order deg(v) + 1.

The Clustering coefficient for an entire graph is defined as:

Definition 8. CC(G)= $\frac{1 * CC(v)}{n}$

Therefore, small world networks usually have a high clustering coefficient, CC(v) and low characteristic path length, L(G).

Math Assignments

1. Proof of the first theorem of graph theory
2. General equations of diameter and radius for the following graphs:
   - Path graph
   - Cycle graph
   - Complete graph
   - Star graph
   - Wheel graph
   - Lollipop graph
3. Investigate random graphs

0.2 June 11th, 2020

Mathematical Induction:
Proof by mathematical induction is a method of proving mathematical formula's and theorems. It involves three main steps.

1. Proving the base case. Eg. n=1
2. Assuming that the theorem is true for a number n=k.
3. Proving that the theorem is true for n=k+1 on the assumption that n=k is true.

Given that the theorem is true for the base case, n=k and n=k+1, then through proof by mathematical induction, the theorem is true for any number within the bounds of the theorem.

Proving the fundamental theorem of graph theory
To prove the fundamental theorem of graph theory, reverse mathematical induction is used.

Let P(n) be the proposition that the sum of the degrees of vertices, v in a graph, G is equal to twice the number of edges, e.

$\sum_v deg(v) = 2 * (\text{ of edges})$

If the e=0, there are no edges and each degree is equal to 0. Therefore P(n) is true for n=0.
Assume that P(n) is true for all graphs with e-1 edges.
Let G have e number of edges, and let uv be an edge on the graph, G. Through the deletion of the vertex uv, we can obtain the graph G-uv. Consequently, the graph G-uv will have e-1 edges. This means that the sum of the degrees on G-uv is 2m-2.
For the graph, G before the deletion of vertex uv, the sum of the degrees will be equal to (2m-2)+2 = 2m. Therefore P(n) is true for n=m.
P(n) is true for n=0, n=m-1 and n=m. Therefore through proof by mathematical induction, P(n) is true for all values of n, $n\epsilon Z^+$.

Effects of Re-scaling on L(G) and CC(G)

Re-scaling a graph means changing the number of vertices.

1. Complete graphs - There is no effect of re-scaling on the L(G) or CC(G). The L(G) and CC(G) of a complete graph will always be equal to one. $L(G) = 1, CC(G) = 1$
2. Path Graphs - $L(G) = \frac{2}{n}$
3. Complete Bipartite Graphs -

0.3 June 12th, 2020

Random Graph Models (presented by Leo)

There are two models of random graphs:

1. $G(n, M)$ model: This model takes two parameter n and M. n denotes the order of the graph and M denotes the size of the graph. The model will randomly choose one graph from the all the possible graphs that satisfy these two parameters.
2. $G(n, p)$ model:This model takes two parameters. Here, n is still the order of the graph, but p represents the probability any edge is connected. The expected number of edges will be $\binom{n}{2}p$. Every graph with n vertex and M edges will have equal probability $p^M(1-p)^{\binom{n}{2}-M}$.

Small World Network paper review (presented by Zahra)

The paper presented novel findings about small world networks. The key findings were that the $L(G)$ of these graphs were quite small and were similar to the $L(G)$ of random graphs. This is due to the fact that it is easy to get from one vertex to another (many connections and vertices). The authors found that $CC(G)$ of the graphs were larger, almost the same magnitude as regular or complete graphs.

The authors also emphasized the ability of diseases to spread faster and at a higher rate in SWN. This is due to the interconnected nature of the nodes and the short paths. This results in requiring very few instances of rewiring / short cuts to result in a SWN.

"Small Worldness" and SWN Indices

Two Main Indices:

1. σ: $\sigma(G) = \frac{CC(G)/CC_R(G)}{L(G)/L_R(G)}$, if $\sigma > 1$, then G is a SWN, where R represents a random graph.

   The disadvantage of this index is that it does not re-scale well to larger networks.
2. ω: ω= $\frac{L_r(G)}{L(G)}$ - $\frac{CC(G)}{CC_l(G)}$, where l represents a lattice graph. If ω is close to -1, it is random. If ω is close to 0, it is a SWN. If ω is close to 1, it is regular.

These indices are used to compare the graph of interest to its random graph in order to determine its "small worldness".

Proof by Graph Induction:

Example 9. Proving the fundamental theorem of Graph Theory.

Proof. Base Case: $|E|$= 1

S=1+1= 2 = 2$|E|$

Induction hypothesis: Suppose true for all k.

Inductive step: Let G be a graph of size k+1, let e ϵ E.

Let G$'$ / e (edge removal), so that G$'$ has size k.

$\Rightarrow$ By hypothesis

S$'$= 2k

S= S$'$ + 2

S= 2k+2

S= 2(k+1)

S=2$|E|$

□

Invariable Topology

Betti Numbers:

- $b_0(G)$= number of connected compartments (or the maximal connected sub-graph)
- $b_1(G)$ number of independent cycles

Theorem 10. *Let G be a graph.* $b_0(G)$ - $b_1(G)$ = $|V|$ - $|E|$

Matrices

Used to look at large networks.

Types of matrices:

1. Adjacency Matrix A(i,j): 1 if edge present, 0 if otherwise
2. Incidence Matrix using e_n and v_n: -1 if edge begins at the vertex, 1 if the edge terminates at this vertex, and 0 if otherwise.
3. Degree Matrix D[i,j]: deg(v_i) if i=j, and 0 if otherwise

Graph Laplacian, Δ(G)

Definition 11. Δ(G)= D(G) - A(G)

Properties: Symmetric, never in-vertible, and $\Delta(G)$ =I(G) * I^T(G)

Laplacian and 2nd Partial Derivatives

Definition 12. Laplace Operator: $\Delta f' = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}$

This is derived below: $\Delta = \nabla \cdot \nabla$

And: $\nabla = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

Therefore: $\Delta f' = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Definition 13. Heat Equation: Models the diffusion of heat from a source, over time. This equation can also correspond to the graph of the heat equation.

$\frac{\partial f}{\partial t}$ = -kΔf, where Δf is the laplacian of f and f is the temperature function.

Making sense of the heat equation

Imagine a point,B on a 1D metal rod with a certain temperature, T_2. If the two adjacent points on either side of the P are on average at a greater temperature than P, then the temperature of P will rise. The opposite is also true.

Let the two adjacent points to P have a temperature of T_1 and T_3 respectively.

Therefore,

If $\frac{T_1+T_3}{2} - T_2 > 0$ then T_2 will rise.

Similarly is $\frac{T_1+T_3}{2} - T_2 < 0$ then T_2 will cool.

The higher the magnitude of $\frac{T_1+T_3}{2} - T_2$, the faster the temperature of T_2 will change.

Therefore,

$\frac{dT_2}{dt} = k(\frac{T_1+T_3}{2} - T_2)$, where k is the proportionality constant.

But,

$\frac{T_1+T_3}{2} - T_2 = (T_3 - T_2) - (T_2 - T_1)$

From above, the following is true,
$\frac{dT_2}{dt} = k(\Delta T_2 - \Delta T_1)$
But, $\Delta T_2 - \Delta T_1$ is the same as $\Delta\Delta T$, or the second difference of T.

This shows that the rate of change of temperature with respect to time is proportional to the second difference of temperature with respect to position.

0.4 June 15th, 2020

Differential Equations

Suppose you want to solve the following pair of differential equations:

$$\frac{du}{dt} = u + v$$
$$\frac{dv}{dt} = u - v$$
$$u(0) = 1,\ v(0) = 2$$

To solve this problem, we can write this system of equations in matrix form.

$$\frac{du}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$
$$\text{If A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

Then one solution of the system of differential equations is shown below.

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{At}\begin{bmatrix} u(0) \\ v(0) \end{bmatrix}$$

The exponential of a matrix

Recall that the value of e^x can be approximated using a Taylor expansion. However, in this case, e^A is a matrix.

$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + ...$

From this, we can see that e^A is a also matrix.

Solving the Graph Heat Equation

$\frac{\partial T}{\partial t} = -k\Delta t$

$T(t) = e^{-k\Delta t} \cdot T(0)$

<u>Note:</u>

For real numbers:
$e^a + e^b = e^{ab}$

For matrices:

$e^a + e^b = e^{ab}$ IFF $AB = BA$

Eigenvalues/Eigenvectors

If we multiply a matrix, A by a vector, $\vec{x}$, the output would be some vector, $\vec{b}$. If the output vector, $\vec{b}$ can be expressed in terms of a constant λ and the input vector $\vec{x}$, then the vector $\vec{x}$ is called an Eigenvector, and the constant λ is known as an Eigenvalue of that Eigenvector.

$$A\vec{x} = \lambda\vec{x}$$

How to compute Eigenvalues and Eigenvectors

To find the Eigenvalue, we use the characteristic polynomial, where A is a given matrix and I is the identity matrix for that order:

$$A - \lambda I$$

By finding the determinant of the characteristic polynomial, and equation it to zero, we can find the Eigenvalue. To find the Eigenvector, substitute the value of λ into the characteristic polynomial, multiply it by a vector $\vec{x}$ and equate this to a vector of 0.

Eigenvectors/values and the Graph Laplacian

- The sum of numbers in a row of the graph laplacian is always equal to zero.
- The graph laplacian is not invertible
- Eigenvalues of the graph laplacian are always non-negative.

0.5 June 17th, 2020

Solution to the heat equation on a graph

The solution to the heat equation on a graph is shown below.

$$\vec{v}(t) = e^{At}\vec{v_0}$$

This can be proved using the taylor expansion.

$$\vec{v}(t) = (I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + ...)\vec{v_0}$$
$$\vec{v}\prime(t) = (A + \frac{2A^2t}{2} + \frac{3A^3t}{3!} + ...)\vec{v_0}$$
$$\vec{v}\prime(t) = A(I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + ...)\vec{v_0}$$
$$\vec{v}\prime(t) = A\vec{v}t$$

Sigma vs Omega Indices

1. σ: $\sigma(G) = \frac{CC(G)/CC_R(G)}{L(G)/L_R(G)}$, if $\sigma > 1$, then G is a SWN, where R represents a random graph.

   However, this index is that it does not re-scale well to larger networks.

2. ω: ω= $\frac{L_r(G)}{L(G)}$ - $\frac{CC(G)}{CC_l(G)}$, where l represents a lattice graph. If ω is close to -1, it is random. If ω is close to 0, it is a SWN. If ω is close to 1, it is regular.

Δ and the Incidence Matrix

There exists two laplacians. The even laplacian, Δ_+, and the odd laplacian, Δ_-. In our calculations, we mostly focus on the even laplacian. The two can be expressed in terms of the incidence matrix, where I^t is the transpose of the incidence matrix.

1. $\Delta_+ = I \cdot I^t$
2. $\Delta_- = I^t \cdot I$

Theorem 14. *For any graph, Δ_+ and Δ_- have the same non-zero eigenvalues.*

With the two forms of the laplacian, we will get two different heat equations. This is because Δ_- depends on the orientation of the graph.

1. The undirected (odd) heat equation: $\frac{\partial\gamma}{\partial t} = -k\Delta_- t$
2. The directed heat equation: $\frac{\partial\gamma}{\partial t} = -k\Delta_+ t$

0.6 June 19th, 2020

Diffusion in a SWN

Consider the differences in the rate of diffusion in a SWN if:

1. The heat was concentrated on a central vertex.
2. The heat was concentrated on a peripheral vertex.

Prove that the rate of diffusion in a SWN will be faster where the heat is concentrated on a central vertex.

Epidemics using the SIR model

The SIR model stands for the number susceptible, infected and recovered people in a population.

Characteristics of the SIR model

1. $S + I + R = N$, where N is the number of people in a network. This model assumes that N will remain constant at any given time.
2. $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$
3. $\frac{dS}{dt} = -bS(t) \cdot I(t)$
4. $\frac{dR}{dt} = R \cdot I(t)$
5. $\frac{dI}{dt} = bS(t)I(t) - kI(t)$

0.7 June 22nd, 2020

Topology on Graphs

Topogoly is study of the shape of an object.

In topology, two shapes are called homotopic if they can be continuously deformed into each other. For example, out of the first five letters of the alphabet, A, B, C, D, E; A,B and D are homotopic while C and E are homotopic.

Topological Invariant

b_0 - The number of connected components

b_1 - The number of independent variables

Theorem 15. $b_0 - b_1 = |V| - |E|$

A tree is a connected graph with no cycles.

Theorem 16. *The fundamental theorem of tree theory:* $|V| = |E| + 1$

Theorem 17. $dim(Ker\Delta_+) = b_0$, $dim(Ker\Delta_-) = b_1$

0.8 June 25th, 2020

Graph Heat Equation:

Remark 18. If the solution at large time converges, then it converges under a different choice of basis.

$\frac{\delta\phi}{\delta t}P = -k\Delta e$
Where, $\Delta = D - V$
From linear algebra:
$[\mathrm{T}]_{st} = [T(e_1)T(e_2)...T(e_n)]$
$\Delta_{SE}= [\Delta(e_1)\Delta(e_2)....\Delta(e_n)]$
Choosing a different basis:
Use the Eigen basis of Δ and the diagonalization method.
If we change from standard to Eigen basis, the formula resembles:
$\frac{\delta\phi\delta t}{=} - kD\phi$
This makes it much easier to solve, using a Taylor expansion.

0.9 June 29th, 2020

Claim:
ker (Δ_-) = span ($\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3$)

Corollary 19. *dim ker* (Δ_-) = *independent cycles*
Δ_- =

2

Comparison of Heat Equations
Odd Heat Equation:
$\frac{\delta\phi}{\delta t} = -k\Delta_-\phi$
vs.
Even heat Equation:
$\frac{\delta\phi}{\delta t} = -k\Delta_+\phi$

0.10 June 30th, 2020

Change of Basis
V vector space (IR^n)
basis α = { e_1, e_2,...,e_n} is standard

T : V $\Rightarrow$ V (linear transformation) $[\mathrm{T}]_\beta^\beta$ $[\mathrm{T}]_\alpha^\alpha$

Theorem 20. *Change of basis:*
$[T]_\beta^\beta = [I]_\beta^\alpha\ [T]_\alpha^\alpha\ [I]_\alpha^\beta$

An observation by Leo:
$[\mathrm{I}]_\beta^\alpha = [\mathrm{I}]_\alpha^\beta$

Linear Algebra on Graphs
T = Laplacian = transfer map
V = space of all possible states

0.11 July 6th, 2020

Markov processes/ chains

What is this?
It is a part of stochastic (random) analysis.

Example 21. Employment versus Unemployment
Every year:

- 0.9 remain employed
- 0.1 become unemployed
- 0.4 remain unemployed
- 0.6 become employed

What happens at the end of time?
Transition matrix (Markov matrix)

M= $\begin{bmatrix} 0.40.6 \\ 0.10.9 \end{bmatrix}$

M $\begin{bmatrix} V_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} V_1 \\ E_1 \end{bmatrix}$

$\mathrm{M}^k \begin{bmatrix} V_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} V_k \\ E_k \end{bmatrix}$

Steady state, as k $\to \infty$

How?
Eigenvalues and eigenvectors

- If M is Markov, 1 is an eigenvalue
- If M is Markov, the eigenvalues are bounded (in absolute value) by 1.

This means at ∞, the only eigenvalue that matters is 1.
Steady state: Eigenvector with eigenvalue 1.
Why? If v is an eigenvector with eigenvalue = 1,
$\mathrm{M}_v = \mathrm{v}$
$\mathrm{M}_v^2 = \mathrm{M}\ (\mathrm{Mv}) = \mathrm{M}_v = \mathrm{v}$
$\mathrm{M}_k = \mathrm{v}$
$\mathrm{M}_v^\infty = \mathrm{v}$

Example 22. Page rank algorithm: Used by search engines, such as Google
Uses eigenvectors to determine the value of a website and rank all websites for a given search.